Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 24, 2002*
Accepted: *July 05, 2002*

Realizations of conformal and Heisenberg algebras in PP-wave-CFT correpondence

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Abstract: We elaborate on the symmetry breaking pattern involved in the Penrose limit of $AdS_{d+1} \times S^{d+1}$ spacetimes and the corresponding limit of the CFT dual. For $d = 2$ we examine in detail how the symmetries contract to products of rotation and Heisenberg algebras, both from the bulk and CFT points of view. Using a free field realization of these algebras acting on products of elementary fields of the CFT with SO(2) R charge +1, we show that this process of contraction restricts all the fields to a few low angular momentum modes and ensures that the field with R charge −1 does not appear. This provides an understanding of several important aspects of the proposal of Berenstein, Maldacena and Nastase. We also indicate how the contraction can be performed on correlation functions.

Keywords: AdS/CFT and dS/CFT Correspondence, Penrose limit and pp-wave background.

Contents

1. Introduction

PP waves provide exact backgrounds for string theory in which the Green-Schwarz world-sheet action becomes quadratic in the lightcone gauge [1]. Such plane waves may be regarded as Penrose limits [2] of $AdS \times S$ spacetimes [3]. Some time ago, Berenstein, Maldacena and Nastase (BMN) [4] used this to propose that the dual description of IIB string theory on the ten-dimensional pp-wave is the large R-charge sector of the $\mathcal{N} = 4$, SU(N) gauge theory. Questions of holography in this background appear to be confusing at this moment and various proposals have been made in [5]–[9]. The proposal has been extended to other related backgrounds [10]. Open strings and D-branes have been studied in this context in [11]. Further insight into the correspondence has been obtained from a semiclassical treatment [12]. Other aspects of string theory in pp-wave background has been studied in [13]. Properties of the Yang-Mills theory relevant to this limit have been studied in [14]. Some questions about black hole formation in these backgrpounds have been addressed in [15].

In [5], the Penrose limit leading to a pp-wave was considered from the point of view of symmetry breaking. The bosonic isometries of $AdS_{d+1} \times S^{d+1}$ backgrounds are $\mathrm{SO}(d,2) \times \mathrm{SO}(d+2)$. In the pp wave limit the number of Killing vectors remain the same and include $\mathrm{SO}(d) \times \mathrm{SO}(d) \times H(d) \times H(d)$ together with translations along the two light cone directions. In terms of the standard metric of a pp-wave

$$ds^2 = 2dx^+dx^- - \mu^2(\vec{x}^2 + \vec{y}^2)(dx^+)^2 + (d\vec{x})^2 + (d\vec{y})^2 \tag{1.1}$$

the first $H(d)$ denotes a Heisenberg algebra which acts on the plane $\vec{x}$ transverse to the wave, while the second $H(d)$ acts on the plane $\vec{y}$. The two $\mathrm{SO}(d)$ factors are rotations in the two transverse planes $\vec{x}$ and $\vec{y}$. (The metric itself has more symmetries, but the RR gauge fields which are necessary for the solution do not). However, the generators of $H(d)$ are *broken* symmetries since they do not commute with the light cone time x^+. Rather, ∂_- is the common central charge for both the Heisenberg algebras while ∂_+ acts as an outer automorphism. The single particle states in this background may be then considered as Nambu-Goldstone bosons. Other discussions of symmetries appear in [16].

In the bulk, supergravity states are created from the light cone vacuum with a given value of the central charge by the action of the creation operators of $H(d) \times H(d)$. This is similar to what happens in $AdS \times S$ where the states are created by the raising operators of the conformal isometries and those of $\mathrm{SO}(d+2)$. For $AdS \times S$, the AdS symmetry generators simplify near the boundary and become differential operators which act entirely on the $S^{d-1} \times (\mathtt{time})$. These then become the conformal symmetries of the holographic theory defined on the boundary, while the $\mathrm{SO}(d+2)$ acts as internal symmetries. These symmetries are in turn used to create states in the CFT which are dual to the supergravity modes.

In the Penrose limit, we have to focus on the *center* of AdS rather than the boundary. In this limit the $AdS \times S$ isometries become the pp-wave isometries which include the Heisenberg algebras. The latter act on the transverse planes while translation in x^+ generate identical copies of the same algebra. In [5] this was used to suggest that there should be a holographic description in which these symmetries are realized on a d-dimensional transverse plane and x^+ acts as a holographic coordinate representing a scale. The other half of these symmetries would be still realized as internal symmetries. Other authors [6] proposed that the holographic theory should involve the entire transverse plane. In [7] the radial coordinate of the transverse plane was proposed as a holographic direction. In [8] it was shown that the boundary of the pp wave spaetime is a one-dimensional null line and it was suggested that this would be the place where a holographic theory should live.

These observations seem to be contradict the original proposal that the dual theory is the same old Yang-Mills theory restricted to the large R-charge sector, since this gauge theory clearly lives on the boundary of $AdS_{d+1} \times S^{d+1}$ which is $S^{d-1} \times \mathtt{time}$. In fact, this boundary is not a part of the pp-wave spacetime. While it is certainly true that this sector of the Yang-Mills reproduces quantities in the string theory, this is not a holographic description in the usual sense.

In this note we shed light on this confusing issue by analyzing how the original symmetries of the theory "contract" in the Penrose limit, both from the bulk point of view as well as from the gauge theory point of view.

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For the simple case of $d = 2$ we first show how Heisenberg algebras arise from the conformal and rotation algebras at the abstract level. We then study in detail a free field realization involving two complex scalar fields. This serves as a toy model for more realistic cases as in e.g. $d = 4$. Starting with a highest weight state created by products of the complex scalar which has SO(2) R charge +1 and creating states by lowering operators of the conformal and rotation groups, we show explicitly how for large conformal weights and R charges the states organize as representations of Heisenberg algebras. We find that at the same time the fields become restricted to a few low angular momentum modes and the scalar with R charge −1 never appears. We indicate how the result would generalize to $AdS_{d+1} \times S^{d+1}$ with $d > 2$, though we do not work out the details. In these cases the reduction of the number of modes is in fact expected to be simpler: the field with SO(2) R charge +1 is restricted to the zero mode and d other lowest angular momentum modes, the fields neutral under the SO(2) are in their zero modes while the field with SO(2) R charge −1 does not participate. These are crucial ingredients of the proposal of BMN where it was argued that operators which contain modes other than those listed above would generically have large anomalous dimensions in large-N. Here we have shown that the same restrictions are required by the contraction of the conformal and rotation algebras to the Heisenberg algebras. Finally we discuss a way to redefine correlation functions in the CFT appropriate to the large weight, large R-charge limit.

2. Isometries in AdS and the AdS/CFT correspondence

Let us recall some aspects of the standard AdS/CFT correspondence [17] relevant to our discussion. We will explicitly deal with $AdS_3 \times S^3$ for simplicity. The results generalize to other dimensions in a straightforward way which we will indicate. In global coordinates the metric is

$$ds^2 = R^2 \left[-(1+\overline{r}^2)d\bar{t}^2 + \frac{d\overline{r}^2}{1+\overline{r}^2} + \overline{r}^2 d\chi^2 + (1-\overline{\rho}^2)d\overline{\theta}^2 + \frac{d\overline{\rho}^2}{1-\overline{\rho}^2} + \overline{\rho}^2 d\phi^2 \right]. \tag{2.1}$$

The isometries are $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ where the two $\mathrm{SL}(2,\mathbb{R})$ factors come from the AdS_3 part and the two SU(2) factors are the standard symmetries of the S^3. We will denote these generators as

$$L_0, L_-, L_+ ; \qquad \overline{L}_0, \overline{L}_-, \overline{L}_+ ; \qquad J_0, J_-, J_+ ; \qquad \overline{J}_0, \overline{J}_-, \overline{J}_+ . \tag{2.2}$$

In terms of null coordinates

$$w = \bar{t} + \chi , \qquad \overline{w} = \bar{t} - \chi \tag{2.3}$$

the L_i are [18]

$$\begin{aligned} L_0 &= i\partial_w , \\ L_- &= ie^{-iw} \left[\frac{2\overline{r}^2+1}{2\overline{r}\sqrt{1+\overline{r}^2}} \partial_w - \frac{1}{2\overline{r}\sqrt{1+\overline{r}^2}} \partial_{\overline{w}} + \frac{i}{2}\sqrt{1+\overline{r}^2}\partial_{\overline{r}} \right], \\ L_+ &= ie^{iw} \left[\frac{2\overline{r}^2+1}{2\overline{r}\sqrt{1+\overline{r}^2}} \partial_w - \frac{1}{2\overline{r}\sqrt{1+\overline{r}^2}} \partial_{\overline{w}} - \frac{i}{2}\sqrt{1+\overline{r}^2}\partial_{\overline{r}} \right]. \end{aligned} \tag{2.4}$$

These satisfy the commutation relations

$$[L_0, L_\pm] = \mp L_\pm\,, \qquad [L_+, L_-] = 2L_0\,. \tag{2.5}$$

The expressions for $\overline{L}_i$ are obtained by interchanging w and $\overline{w}$. The expressions for J_i and $\overline{J}_i$ can be similalrly written down by analytically continuing these expressions. We will not need these, since (2.4) would be sufficient to make the point.

The states of some field in the bulk form some representation of the algebra. Consider for example a massless scalar field in six dimensions. The lowest energy state is a highest weight state which satisfies

$$L_+ \,|h\rangle = \overline{L}_+ \,|h\rangle = 0\,, \qquad L_0 \,|h\rangle = \overline{L}_0 \,|h\rangle = h \,|h\rangle\,. \tag{2.6}$$

The weight h is determined in terms of the $\mathrm{SO}(4) = \mathrm{SU}(2)\times\mathrm{SU}(2)$ representation content. The creation of states in each represntation is standard and will not be repeated here. If the $\mathrm{SO}(4)$ angular momentum is L, so that the quadratic Casimir is $L(L+2)$ one has

$$2h = L + 2\,. \tag{2.7}$$

The descendants may be obtained by the action of the L_-'s

$$|n, \overline{n}, h\rangle = (L_-)^n (L_-)^{\overline{n}} \,|h\rangle\,. \tag{2.8}$$

This has

$$L_0 \,|n,\overline{n},h\rangle = (h+n)\,|n,\overline{n},h\rangle\,, \qquad \overline{L}_0 \,|n,\overline{n},h\rangle = (h+\overline{n})\,|n,\overline{n},h\rangle\,. \tag{2.9}$$

As is clear from the expressions for the L_0 and $\overline{L}_0$, the total energy ω is the value of $L_0 + \overline{L}_0$ while the angular momentum l along the circle whose coordinate is χ is the value of $\overline{L}_0 - L_0$. Thus we have

$$\omega = 2h + n + \overline{n}\,, \qquad l = \overline{n} - n\,. \tag{2.10}$$

It will be useful to relate this discussion to the normalizable solutions of the wave equation. The solution for the quantum numbers given above is given by

$$\begin{aligned}\Psi(\overline{t},\overline{r},\chi;\Omega) \,&=\, e^{-i\omega\overline{t}} Y_L(\Omega)\; e^{il\chi} \left[\frac{1}{\sqrt{1+\overline{r}^2}}\right]^{2h} \left[\frac{\overline{r}}{\sqrt{1+\overline{r}^2}}\right]^{l} \times \\ &\quad\times F\left(2h+n+l, -n; 2h; \frac{1}{\sqrt{1+\overline{r}^2}}\right).\end{aligned} \tag{2.11}$$

Using the properties of hypergeometric functions this may be rewritten as

$$\begin{aligned}\Psi(\overline{t},\overline{r},\chi;\Omega) \,&=\, e^{-i\omega\overline{t}} Y_L(\Omega)\; e^{il\chi} \left[\frac{1}{\sqrt{1+\overline{r}^2}}\right]^{2h} \left[\frac{\overline{r}}{\sqrt{1+\overline{r}^2}}\right]^{l} \times \\ &\quad\times F\left(2h+n+l, -n; l+1; \frac{\overline{r}^2}{\sqrt{1+\overline{r}^2}}\right).\end{aligned} \tag{2.12}$$

The key property which leads to a holographic description on the boundary at $\overline{r} = \overline{r}_0 \to \infty$ is that these wave functions have a universal behavior for large $\overline{r}$. It is clear from (2.11) that $\Psi \sim (\overline{r}_0)^{-2h}$ regardless of the values of n and l. This means that a local bulk operator becomes a local boundary operator up to an overall factor of the cutoff $\overline{r}_0$. In the holographic description it is this boundary operator which creates the state and $\overline{r}_0$ becomes a scale in the theory.

This behavior of the wavefunction also reduces the generators of L_i and $\overline{L}_i$ to standard forms. On the boundary one gets

$$\begin{aligned} L_0 &= i\partial_w\,, \\ L_- &= ie^{-iw}[\partial_w - ih]\,, \\ L_+ &= ie^{iw}[\partial_w + ih] \end{aligned} \tag{2.13}$$

which are the standard form of $\mathrm{SL}(2,\mathbb{R})$ generators acting on a primary field of weight h. There are similar expressions for $\overline{L}_i$. Thus the states can be created in the CFT side in a fashion identical to that in the bulk. The difference is that now the wavefunctions are functions of $w, \overline{w}$, i.e. in a $1+1$-dimensional theory on the boundary. The generators of the S^3 isometries remain the same as we approach the boundary since they do not involve $\overline{r}$. These symmetries are realized as internal symmetries in the CFT. This makes it clear why the natural location of the holographic theory is on the boundary.

The above discussion may be easily generalized to higher dimensions. Representing AdS_{d+1} by an equation

$$X_1^2 + X_2^2 - X_3^2 - \cdots X_{d+2}^2 = R^2 \tag{2.14}$$

in a flat $d+2$-dimensional space with signature $(-1,-1,1,1,\ldots)$ the conformal isometries are rotations in this embedding space, denoted by J_{AB}. Writing the AdS_{d+1} metric as

$$ds^2 = R^2\left[-(1+\overline{r}^2)d\overline{t}^2 + \frac{d\overline{r}^2}{1+\overline{r}^2} + \overline{r}^2 d\Omega^2\right], \tag{2.15}$$

where $d\Omega^2$ is the standard metric on a $d-1$-dimensional sphere, it is clear that the generators $J_{ij}, i, j = 3\cdots d+2$ are the $\mathrm{SO}(d)$ rotations on this sphere. The energy is given by the generator J_{12}. These are then the generalizations of the generators L_0 and $\overline{L}_0$ of the three-dimensional case. The remaining generators J_{1i} and J_{2i} are the conformal symmetries which are the generalizations of $L_\pm$ and $\overline{L}_\pm$. Once again these become the standard generators of conformal symmetries on the boundary.

3. The Penrose limit and PP waves

To get to the six-dimensional pp-wave from $AdS_3 \times S^3$, we first define

$$\begin{aligned} t &= \overline{t}\cosh\alpha - \overline{\theta}\sinh\alpha\,, \\ \theta &= \overline{t}\sinh\alpha + \overline{\theta}\cosh\alpha\,. \end{aligned} \tag{3.1}$$

Then we rescale

$$r = R\overline{r}\,, \qquad \rho = \bar{\rho}\,, \qquad x^\pm = \frac{R}{\sqrt{2}}(\theta \pm t) \tag{3.2}$$

and take the limit

$$R, \alpha \longrightarrow \infty\,, \qquad r, \rho, x^{\pm} = \text{fixed}\,, \qquad \mu = \frac{e^{\alpha}}{\sqrt{2}\;R} = \text{fixed}\,. \tag{3.3}$$

In this limit the metric reduces to (1.1) with the definitions

$$\begin{aligned} \vec{x} &= (x_1, x_2)\,, \qquad x_1 = r\cos\chi\,, \qquad x_2 = r\sin\chi\,, \\ \vec{y} &= (y_1, y_2)\,, \qquad y_1 = \rho\cos\phi\,, \qquad y_2 = \rho\sin\phi\,. \end{aligned} \tag{3.4}$$

The parameter μ may be set to unity (if non zero) by rescaling $x^{\pm}$. In the following we will set $\mu = 1$. These formulae can be trivially generalized to higher dimensions.

We want to see what happens to the symmetry generators of $AdS_3 \times S^3$ in this limit. This means — among other things — that we have to take the $\mathrm{SL}(2,\mathbb{R})$ generators given in (2.4) and focus on the region $\overline{r} \to 0$ keeping r defined above fixed. This is the opposite of the limit taken to reduce the isometries of the bulk to conformal symmetries on the boundary. Performing the Penrose limit as described above we get the following limiting form of the generators

$$\begin{aligned} L_0 + \overline{L}_0 &= i\partial_+ - 2iR^2\partial_-\,, \\ L_0 - \overline{L}_0 &= i\partial_\chi\,, \\ L_- + \overline{L}_- &= Re^{-ix^+}\left[\frac{\partial}{\partial x_1} - ix_1\partial_-\right] \equiv Ra_1^\dagger\,, \\ L_- - \overline{L}_- &= iRe^{-ix^+}\left[\frac{\partial}{\partial x_2} - ix_2\partial_-\right] \equiv Ra_2^\dagger\,, \\ L_+ + \overline{L}_+ &= Re^{ix^+}\left[\frac{\partial}{\partial x_1} + ix_1\partial_-\right] \equiv Ra_1\,, \\ L_- + \overline{L}_- &= Re^{ix^+}\left[\frac{\partial}{\partial x_2} + ix_2\partial_-\right] \equiv Ra_2\,. \end{aligned} \tag{3.5}$$

In this limit the algebra becomes

$$[a_i, a_j^\dagger] = -2i\delta_{ij}\partial_- \tag{3.6}$$

while

$$[L_0 - \overline{L}_0, a_1] = a_2\,, \qquad [L_0 - \overline{L}_0, a_2] = -a_1\,. \tag{3.7}$$

The generator ∂_- commutes with every other generator and acts as a central charge. To leading order this is essentially $L_0 + \overline{L}_0$. The $a_j, a_j^\dagger$ then form a Heisenberg algebra. The limiting form of the generators in (3.5) are well known symmetries of the pp-wave background [16].

From the isometries of S^3 we get a similar structure, viz two more sets of oscillators $b_1, b_1^\dagger, b_2, b_2^\dagger$ which may be obtained by replacing x_j in (3.5) by the y_j. In addition we have

$$\begin{aligned} J_0 + \overline{J}_0 &= -i\partial_+ - iR^2\partial_-\,, \\ J_0 - \overline{J}_0 &= i\partial_\phi\,. \end{aligned} \tag{3.8}$$

Significantly, the central charge which appears in the oscillator algebra b_j is the same as that in the a_j algebra.

The element ∂_+ acts as an outer automorphism, generating identical copies of the algebra at different values of x^+. From the expressions above we have

$$\begin{aligned} L_0 + \overline{L}_0 - J_0 - \overline{J}_0 &= i\partial_+ \,, \\ L_0 + \overline{L}_0 + J_0 + \overline{J}_0 &= -iR^2\partial_- \,. \end{aligned} \tag{3.9}$$

To obtain the supergravity states of the bulk theory we start with the "light cone vacuum" specified by the value of the central charge $-i\partial_- = p_-$ which is annihilated by all the destruction operators a_j, b_j. This is the lowest energy state. The higher states are created as usual by action of the creation operators just as in a multidimensional harmonic oscillator. Thus one has states of the form

$$|n_i, m_a, p_-\rangle = \prod_i (a_i^\dagger)^{n_i} \prod_a (b_a^\dagger)^{m_a} \; |0, p_-\rangle \,. \tag{3.10}$$

The light cone energy is given by

$$p^- \equiv p_+ = \sum_j n_j + \sum_a m_a + 2 \,. \tag{3.11}$$

The wave functions are given by standard Hermite polynomials in x_j, y_a.

It is interesting to see how the AdS wavefunctions become these wave functions. To do that it is easier to work in the coordinates r, ρ, χ, ϕ. Consider a massless scalar in six dimensions. The wavefunctions are then

$$\Psi = e^{-ip_+x^+ + ip_-x^-} \, e^{il\chi + ij\phi} e^{-\frac{1}{2}p_-(r^2+\rho^2)} \; L_n^l(p_-r^2) \; L_m^j(p_-\rho^2) \,, \tag{3.12}$$

where L denotes a Laguerre function. The dispersion relation is then

$$p_+ = 2n + l + 2m + j + 2 \,. \tag{3.13}$$

Equation (3.12) should be compared with (2.12) in the Penrose limit.[1] First consider the radial part of (2.12). For large angular momentum on the S^3 we have $2h \sim p_-R^2$. Near $\overline{r} = 0$ with $r = R\overline{r}$ fixed the first two factors become

$$\left[\frac{1}{\sqrt{1+\overline{r}^2}}\right]^{2h} \left[\frac{\overline{r}}{\sqrt{1+\overline{r}^2}}\right]^{l} \sim e^{-\frac{1}{2}p_-r^2} \; r^l \tag{3.14}$$

while the hypergeometric function simplifies to[2]

$$F\left(p_-R^2, -n; l+1; \overline{r}^2\right) \sim L_n^l(p_-r^2) \,. \tag{3.15}$$

The expressions (3.15) and (3.14) then gives the r dependent part of (3.12). The spherical

[1] A similar comparison was performed in [7], however the wavefunctions used in this paper did not contain the Laguerre polynomial piece.

[2] Note that since the third argument of the hypergeometric function is a negative integer, it is a polynomial rather than an infinite series.

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harmonic Y_L may be written in the form

$$
\begin{aligned}
Y_L(\Omega) = {} & e^{iJ\overline{\theta}} e^{ij\phi} \left[\frac{1}{\sqrt{1-\overline{\rho}^2}}\right]^{-2h} \left[\frac{\overline{\rho}}{\sqrt{1-\overline{\rho}^2}}\right]^{j} \times \\
& \times F\left(2h+m+l, -m; j+1; \frac{\overline{\rho}^2}{\sqrt{1-\overline{\rho}^2}}\right). \qquad (3.16)
\end{aligned}
$$

This leads to the ρ dependent part of (3.12). Finally noting that $J = R^2 p_-$ the $\overline{t}$ and $\overline{\theta}$ factors of (3.16) and (2.12) combine to give the $x^{\pm}$ dependent phases of (3.12).

Once again it is straightforward to generalize the discussion to higher dimensions. As explained at the end of the previous section the generators J_{ij} of the AdS part of the isometries remain as they are and become the SO(d) generators acting on the $\vec{x}$ part of the transverse plane. To leading order, the generator $J_{12} = \partial_{\overline{t}}$ becomes a central term, and the generators J_{1i} and J_{2i} combine to form the Heisenberg algebra on the $\vec{x}$ plane. On the S side, the story is similar. The SO(d) rotations simply carry over. The other off-diagonal generators reduce to Heisenberg algebra generators acting on the transverse plane $\vec{y}$. The generator $\partial_{\overline{\theta}}$ becomes a central term and is equal to $\partial_{\overline{t}}$ to leading order. The difference between $\partial_{\overline{t}}$ and $\partial_{\overline{\theta}}$ is however finite and provides the outer automorphism ∂_+ while the sum provides the common central term of the two Heisenberg algebras. The reduction of the wavefunctions to the correct pp-wave wavefunctions also follow.

The generators (3.5) involve derivatives with respect to the radial coordinates r and ρ in an essential way. There is no simple way in which these are related to the way the conformal algebra acts on the dual theory via the generators in (2.13) simply because the Penrose limit involved going to the $\overline{r} = 0$ region. From this point of view it appears mysterious how one would realize the algebra of symmetries of the pp-wave in the dual gauge theory which lives in a region which is *not contained in the pp-wave geometry.*

4. From conformal and rotation algebras to Heisenberg algebras

Before delving into the question how the pp-wave symmetries are realized in the dual theory let us examine how the contraction of conformal and rotation algebras happens at the abstract level. Specifically we will show that the conformal algebra reduces to a product of rotation algebra and a Heisenberg algebra when we act on states with large conformal weight. Similarly, when a rotation algebra acts on states of large angular momentum, it reduces to a product of a lower-dimensional rotation algebra and a Heisenberg algebra. Finally we will combine the original conformal and rotation algebras. We will discuss the case of SO$(2,2)\times$SO(4) in detail. The result generalizes easily to other SO$(d,2)\times$SO$(d+2)$.

4.1 Conformal algebra $\to$ Heisenberg algebra

Each of the SL$(2,\mathbb{R})$ factors of SO$(2,2)$ satisfies the algebra (2.5). We will consider a highest weight state $|h,\overline{h};0,0\rangle$

$$
\begin{aligned}
& L_+\,|h,\overline{h};0,0\rangle = \overline{L}_+\,|h,\overline{h};0,0\rangle = 0\,, \qquad L_0\,|h,\overline{h},0,0\rangle = h\,|h,\overline{h};0,0\rangle\,, \\
& \overline{L}_0\,|h,\overline{h},0,0\rangle = \overline{h}\,|h,\overline{h},0,0\rangle \qquad (4.1)
\end{aligned}
$$

and the descendants of level $(n,\overline{n})$

$$|h,\overline{h},n,\overline{n}\rangle = (L_-)^n(\overline{L}_-)^{\overline{n}}\ |h,\overline{h},0,0\rangle\,. \tag{4.2}$$

Consider first the action of one of the $\mathrm{SL}(2,\mathbb{R})$'s. The algebra implies

$$\begin{aligned} L_0\ |h,\overline{h},n,\overline{n}\rangle &= (h+n)\ |h,\overline{h},n,\overline{n}\rangle\,, \qquad L_-|h,\overline{h},n,\overline{n}\rangle = |h,\overline{h},n+1,\overline{n}\rangle\,,\\ L_+\ |h,\overline{h},n,\overline{n}\rangle &= [2nh+n(n-1)]\ |h,\overline{h},n-1,\overline{n}\rangle\,. \end{aligned} \tag{4.3}$$

It is now clear that when $n \ll h$ the generator L_0 becomes a c-number, and

$$L_-\ |h,\overline{h},n,\overline{n}\rangle = |h,\overline{h},n+1,\overline{n}\rangle\,, \qquad L_+\ |h,\overline{h},n,\overline{n}\rangle = 2nh\ |h,\overline{h},n-1,\overline{n}\rangle\,. \tag{4.4}$$

These are precisely the relations obtained from the action of annihilation and creation operators of a Heisenberg algebra. Thus L_+ and L_- may be regarded as annihilation and creation operators A and $A^\dagger$, respectively, with

$$[A,A^\dagger] = 2h\,. \tag{4.5}$$

An identical contraction of course happens for the other $\mathrm{SL}(2,\mathbb{R})$ piece. Acting by generators $\overline{L}_-$ we get states which are specified by another level number $\overline{n}$ and we have another Heisenberg algebra $\overline{A},\overline{A}^\dagger$ which commute to $2\overline{h}$. The sum and difference $L_+ \pm \overline{L}_+$ behave as two independent annihilation operators while $L_- \pm \overline{L}_-$ behave as creation operators and these commute to $2(h+\overline{h})$. The *difference* $L_0 - \overline{L}_0$ usual the angular momentum of the state. Thus we have an algebra $\mathrm{SO}(2)\times H(2)$ together with a central charge.

The above discussion may be easily generalized to $\mathrm{SO}(d,2)$. The role of $L_\pm, \overline{L}_\pm$'s is now replaced by the d sets of raising and lowering operators. The role of $L_0+\overline{L}_0$ is replaced by dilatation, while $L_0-\overline{L}_0$ is replaced by the $d(d-1)/2$ angular momentum generators.

4.2 Rotation algebra $\rightarrow$ Heisenberg algebra

The isometries of S^3 also reduce to $\mathrm{SO}(2)\times H(2)$. The isometries of S^3 are $\mathrm{SU}(2)\times\mathrm{SU}(2)$ with generators $J_0, J_\pm$ and $\overline{J}_0, \overline{J}_\pm$, respectively, with the usual algebra

$$[J_+,J_-] = 2J_0\,, \qquad [J_0,J_\pm] = \pm J_\pm \tag{4.6}$$

and similarly for the $\overline{J}$'s. Starting with a highest weight state $|j,\overline{j};0,0\rangle$

$$\begin{aligned} J_+\ |j,\overline{j};0,0\rangle &= \overline{J}_+\ |j,\overline{j};0,0\rangle = 0\,,\\ J_0\ |j,\overline{j};0,0\rangle &= j\ |j,\overline{j};0,0\rangle\,, \qquad \overline{J}_0\ |j,\overline{j};0,0\rangle = \overline{j}\ |j,\overline{j};0,0\rangle\,. \end{aligned} \tag{4.7}$$

We create the other states in this representation as usual

$$|j,\overline{j},m,\overline{m}\rangle = (J_-)^m(\overline{J}_-)^{\overline{m}}\ |j,\overline{j};m,\overline{m}\rangle\,. \tag{4.8}$$

We are using an unconventional normalization for conveninence. In this case the representation is finite dimensional, so that the maximum values of $m,\overline{m}$ can be $2j, 2\overline{j}$, respectively.

Using the algebra we have well known relations

$$\begin{aligned}
J_0 \, |j,\bar{j};m,\bar{m}\rangle &= (j-m) \, |j,\bar{j},m,\bar{m}\rangle \,, \qquad \bar{J}_0 \, |j,\bar{j};m,\bar{m}\rangle = (\bar{j}-\bar{m}) \, |j,\bar{j},m,\bar{m}\rangle \,, \\
J_- \, |j,\bar{j};m,\bar{m}\rangle &= |j,\bar{j};m+1,\bar{m}\rangle \,, \qquad \bar{J}_- \, |j,\bar{j};m,\bar{m}\rangle = \, |j,\bar{j};m,\bar{m}+1\rangle \,, \\
J_+ \, |j,\bar{j};m,\bar{m}\rangle &= [2mj - m(m-1)] \, |j,\bar{j};m-1,\bar{m}\rangle \,, \\
\bar{J}_+ \, |j,\bar{j};m,\bar{m}\rangle &= [2\bar{m}\bar{j} - \bar{m}(\bar{m}-1)] \, |j,\bar{j};m,\bar{m}-1\rangle \,.
\end{aligned} \tag{4.9}$$

When $j, \bar{j}$ are large and $m \ll j$, $\bar{m} \ll \bar{j}$ we can ignore the fact that the representations have finite dimensionalities $(2j+1, 2\bar{j}+1)$. Furthermore, $J_-, \bar{J}_-$ and $J_+, \bar{J}_+$ become the creation and annihilation operators of two commuting Heisenberg algebras with planck's constants $2j$ and $2\bar{j}$, respectively. Taking sums and differences of $J_\pm$ and $\bar{J}_\pm$ we get a common central charge for both the Heisenberg algebras which is $2(j+\bar{j})$, while the difference $(j-\bar{j})$ is a SO(2) angular momentum.

Once again the above considerations generalize to higher-dimensional rotation group SO$(d+2)$ leading to SO$(d) \times H(d)$.

4.3 Combining the conformal and rotation algebras

In the Penrose limit we are interested in states which have both large conformal dimensions as well as large SO$(d+2)$ angular momenta. For example in $d=2$ we are interested in states which are constructed out of highest weight states of the form $|h, \bar{h}; 0, 0\rangle \otimes |j, \bar{j}; 0, 0\rangle$ with

$$h + \bar{h} = j + \bar{j} \,. \tag{4.10}$$

In a supersymmetric theory these states have a special significance since these are chiral primaries. The descendants of this state are

$$(L_-)^n \, (\bar{L}_-)^{\bar{n}} \, (J_-)^m \, (\bar{J})^{\bar{m}} \, |h, \bar{h}; 0, 0\rangle \otimes |j, \bar{j}; 0, 0\rangle \,. \tag{4.11}$$

Acting on the module, the product of conformal and rotation algebras contract to SO$(2) \times$ SO$(2) \times H(2) \times H(2)$ and the central charges of the two $H(2)$'s are the same. To leading order this common central charge is the value of the sum $\frac{1}{2}(L_0 + \bar{L}_0 + J_0 + \bar{J}_0)$. The *difference* $(L_0 + \bar{L}_0) - (J_0 + \bar{J}_0)$ provides an outer automorphism of the Heisenberg algebras. This is exactly what happens for the isometries of the pp-wave.

5. Heisenberg algebras in the CFT

The discussion above has no reference to any specific way in which the symmetries are realized. We have seen in the section 3 that when these are realized as isometries of $AdS_3 \times S^3$ this contraction corresponds to the Penrose limit which lead us to the isometries of the pp-wave background. We now want to see how this mechanism happens in the CFT description.

In the CFT, the states form representations of the conformal and R symmetry algebras. The pp-wave/CFT correspondence requires that a subset of these states should survive in the large conformal weight and large R charge limit, and should form representations of a product of lower-dimensional rotation and Heisenberg algebras. In this section we will study this in a free field realization and show that this requirement implies that only a few of the original angular momentum modes survive in this limit. We will perform the analysis for $AdS_3 \times S^3$ and indicate how the result generalizes to higher dimensions

The free field realization consists of four scalar fields living on $S^1 \times \mathtt{time}$ which we will organize into two complex scalars U, V with their complex conjugates $\overline{U}, \overline{V}$. They together form a vector representation of SO(4). We will pick a particular SO(2) subgroup of SO(4) such that U has R charge $+1$, $\overline{U}$ has charge -1 and $V, \overline{V}$ are neutral.

Each of the fields have a mode expansion

$$\begin{aligned} U &= \frac{i}{\sqrt{4\pi}} \sum_n \frac{1}{n} \left[a_n e^{-inw} + \overline{a}_n e^{-in\overline{w}} \right], \\ \overline{U} &= \frac{i}{\sqrt{4\pi}} \sum_n \frac{1}{n} \left[a^\star{}_n e^{-inw} + \overline{a^\star}_n e^{-in\overline{w}} \right], \\ V &= \frac{i}{\sqrt{4\pi}} \sum_n \frac{1}{n} \left[b_n e^{-inw} + \overline{b}_n e^{-in\overline{w}} \right], \\ \overline{V} &= \frac{i}{\sqrt{4\pi}} \sum_n \frac{1}{n} \left[b^\star{}_n e^{-inw} + \overline{b^\star}_n e^{-in\overline{w}} \right], \end{aligned} \tag{5.1}$$

where the non-vanishing commutators are

$$[a_n, a^\star{}_m] = [b_n, b^\star{}_m] = \left[\overline{a}_n, \overline{a^\star}_m\right] = \left[\overline{b}_n, \overline{b^\star}_m\right] = 2n\delta_{m+n,0}\,. \tag{5.2}$$

In terms of these modes the generators $L_i, \overline{L}_i$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ are given by

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \left(a_{n-m} a^\star{}_m + b_{n-m} b^\star{}_m \right), \qquad (n = 0, \pm 1) \tag{5.3}$$

while the generators $J_i, \overline{J}_i$ of the $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{SO}(4)$ are given by

$$\begin{aligned} J_0 &= \frac{1}{4} \int d\phi \left[U \frac{\delta}{\delta U} - \overline{U} \frac{\delta}{\delta \overline{U}} - V \frac{\delta}{\delta V} + \overline{V} \frac{\delta}{\delta \overline{V}} \right], \\ J_+ &= \frac{1}{2} \int d\phi \left[\overline{V} \frac{\delta}{\delta \overline{U}} - U \frac{\delta}{\delta V} \right], \\ J_- &= \frac{1}{2} \int d\phi \left[\overline{U} \frac{\delta}{\delta \overline{V}} - V \frac{\delta}{\delta U} \right], \\ \overline{J}_0 &= \frac{1}{4} \int d\phi \left[U \frac{\delta}{\delta U} - \overline{U} \frac{\delta}{\delta \overline{U}} + V \frac{\delta}{\delta V} - \overline{V} \frac{\delta}{\delta \overline{V}} \right], \\ \overline{J}_+ &= \frac{1}{2} \int d\phi \left[V \frac{\delta}{\delta \overline{U}} - U \frac{\delta}{\delta \overline{V}} \right], \\ \overline{J}_- &= \frac{1}{2} \int d\phi \left[\overline{U} \frac{\delta}{\delta V} - \overline{V} \frac{\delta}{\delta U} \right]. \end{aligned} \tag{5.4}$$

The SO(2) in question is generated by $J_0 + \overline{J}_0$. These may be expressed in terms of modes using the mode expansions above. All operators are assumed to be normal ordered.

The effect of the various $\mathrm{SL}(2,\mathbb{R})$ generators on primary fields is given by derivative operators as described in eq. (2.13), while that the SU(2) generators rotate the fields among them.

To discuss states it is useful to euclideanize the time $\overline{t}$ by defining $\tau = -i\overline{t}$ and perform a conformal transformation to R^2 with complex coordinates $z, \overline{z}$

$$z = e^{\tau + i\chi}\,, \qquad \overline{z} = e^{\tau - i\chi}\,. \tag{5.5}$$

On the plane the action of L_i become

$$\begin{aligned} L_0 &= -(z\partial_z + h)\,, \qquad \overline{L}_0 = -(\overline{z}\partial_{\overline{z}} + \overline{h})\,, \\ L_- &= -\partial_z\,, \qquad \overline{L}_- = -\partial_{\overline{z}}\,, \\ L_+ &= -(z^2\partial_z + 2hz)\,, \qquad L_+ = -(\overline{z}^2\partial_{\overline{z}} + 2\overline{h}\overline{z})\,. \end{aligned} \tag{5.6}$$

The $\overline{L}$'s are obtained by replacing z with $\overline{z}$.

Consider the operator on the plane

$$\mathcal{O}(z,\overline{z}) = [\partial_z U(z)]^j [\partial_{\overline{z}} U(\overline{z})]^j\,. \tag{5.7}$$

At $z = \overline{z} = 0$ this creates a highest weight state which we will denote by

$$|j;0,0;0,0\rangle = \mathcal{O}(0,0)|0\rangle\,. \tag{5.8}$$

The first slot simply labels the representation, the next two zeros denote that they are at level $(0,0)$ of the conformal algebra and at level $(0,0)$ of the rotation algebra. (5.8) is clearly a highest weight state, killed by $L_+, \overline{L}_+, J_+, \overline{J}_+$. Furthermore

$$\begin{aligned} L_0|j;0,0;0,0\rangle &= \overline{L}_0|j;0,0;0,0\rangle = J_0|j;0,0;0,0\rangle \\ &= \overline{J}_0|j;0,0;0,0\rangle = j\ |j;0,0;0,0\rangle\,. \end{aligned} \tag{5.9}$$

In terms of the modes this state is

$$|j;0,0;0,0\rangle = (a_{-1})^j(\overline{a}_{-1})^j|0\rangle\,. \tag{5.10}$$

The descendants of this state are of the form

$$|j;n,\overline{n};m,\overline{m}\rangle = (L_-)^n(\overline{L}_-)^{\overline{n}}(J_-)^m(\overline{J}_-)^{\overline{m}}|j;0,0;0,0\rangle\,. \tag{5.11}$$

In terms of the operator creating the state, action of L_- is simply a derivative ∂_z while J_- replaces one of the fields U by some other field.

From the discussion of the abstract algebras we know that

$$\begin{aligned} L_+|j;n,\overline{n};m,\overline{m}\rangle &= [2jn + n(n-1)]\ j;n-1,\overline{n};m,\overline{m}\rangle\,, \\ J_+|j;n,\overline{n};m,\overline{m}\rangle &= [2jm - m(m-1)]\ |j;n,\overline{n};m,\overline{m}\rangle \end{aligned} \tag{5.12}$$

and similarly for the action of $\overline{L}_+, \overline{J}_+$. In the large j and $n, \overline{n}, m, \overline{m} \ll j$ limit these relations characterise Heisenberg algebras. We want to see what is involved in this contraction.

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5.1 The conformal part

Let us first discuss the conformal descendants. It is sufficient to deal with one of the $SL(2,\mathbb{R})$ factors. The simplest non-trivial example is

$$|j;2,0;0,0\rangle = \left[j(j-1)(a_{-2})^2(a_{-1})^{j-2} + 2j(a_{-3})(a_{-1})^{j-1}\right]\ (\overline{a}_{-1})^j|j;0,0\rangle\,. \tag{5.13}$$

The operator which creates this state is

$$\partial_z^2\mathcal{O}(z) = \left[j(j-1)(\partial_z^2U)^2(\partial_zU(z))^{j-2} + 2j(\partial_z^3U)(\partial_zU(z))^{j-1}\right](\partial_{\overline{z}}U)^j \tag{5.14}$$

the first term in (5.13) comes from the first term in (5.14) and similarly for the second term. Even at this stage, it is clear that the terms which involve ∂^3 is subdominant in the large-j limit. Let us now check the action of L_+ on this state. This may be easily calculated to yield

$$L_+\ |j;2,0;0,0\rangle = [4(j-1)+6]\ |j;1,0;0,0\rangle \tag{5.15}$$

which is consistent with the general result of (4.3).

The first term of the right hand side of (5.15) comes from the action of L_+ on the first term of (5.13), viz. the state $[j(j-1)(\alpha_{-2})^2(\alpha_{-1})^{j-2}\ |0\rangle$. In the large-$j$ small n limit only the first term in (5.15) contributes and one has

$$L_+\ |j;2,0;0,0\rangle \simeq 4j\ |j;1,0;0,0\rangle \tag{5.16}$$

which, as argued above, is characteristic of Heisenberg algebra.

It is easy to generalize this result for arbitrary n. ∂_z^n distributes among the product of (∂_zU)'s. Ignoring the factors of $\partial_{\overline{z}}U$ which are always present,

$$\partial_z^n\mathcal{O}(z) = [j(j-1)\cdots(j-n+1)]\,(\partial(\partial_zU))^n(\partial_zU)^{j-n} + \cdots\,. \tag{5.17}$$

The elipses denote terms of the form $(\partial(\partial_zU))^{n-2}(\partial^2(\partial_zU))(\partial_zU)^{j-n+1}$ and those with more derivatives on a single (∂_zU). Note that the first term has n powers of j while the others have lower powers of j. It may be easily seen that the state $|j:n,0;0,0\rangle$ is of the form

$$|j;n,0;0,0\rangle = [j(j-1)\cdots(j-n+1)]\,(a_{-2})^n\ (a_{-1})^{j-n}\ |0\rangle + O(j^{n-1})\,. \tag{5.18}$$

Once again the leading term in $L_+\ |j;n,0;0,0\rangle$ comes from this first term in (5.18) and leads to

$$L_+\ |j;n,0;0,0\rangle \simeq 2nj\ |j;n-1,0;0,0\rangle \tag{5.19}$$

with subleading contributions which do not involve j.

It is now clear what is happening in the contraction of the conformal algebra to the Heisenberg algebra. The raising operator L_- acts as a derivative on the complex plane. The only term which is relevant in the action of L_-^n on $(\partial_zU)^j$ is the term where we have n products of (∂_z^2U) and none involving higher derivatives on U. In terms of modes, this means that the operators involve only a_{-1} and a_{-2}. In the theory on $S^1\times$ `time` these are the modes with angular momentum $l=1$ and $l=2$ on the S^1. Operators constructed in this fashion automatically furnish a representation of the Heisenberg algebra with a "Planck's constant" equal to $2j$.

It should be be possible to extend the above argument easily to higher dimensions. For a CFT living in euclidean R^d have d raising operators in the conformal algebra. We start with some elementary field U with dimension 1 and construct a highest weight state by action of operators like $(U)^j$. Descendants are obtained by action of these raising operators which act like derivatives. In the large-j limit the only terms which survive in these descendants are the ones which contain single derivatives. There is one crucial difference from the two-dimensional case discussed above. Now the vacuum is not annihilated by the zero mode of (U). Thus the highest weight state has zero angular momentum but non-zero energy (equal to j) and is created by products of the zero mode of (U). The descendants have higher angular momenta. However the above discussion shows that in the large-j limit only the d states of lowest angular momentum, $l = 1$ participate. This is precisely the proposal of BMN and plays an important role in the dynamics at large R-charge [14]. We have obtained this from the contraction of the conformal algebra to the Heisenberg algebra.

5.2 The R symmetry part

The action of J_- replaces one of the U's in the highest weight state by a V

$$[J_-, \mathcal{O}] \sim j \;\left[(\partial_z V)(\partial_z U)^{j-1}(\partial_{\bar{z}} U)^j + (\partial_{\bar{z}} V)(\partial_z U)^j(\partial_{\bar{z}} U)^{j-1}\right]. \tag{5.20}$$

The first level descendant of the R symmetry algebra is

$$\begin{aligned} |j;0,0;1,0\rangle &= J_-|j;0,0;0,0\rangle \\ &= -\frac{j}{2}\;\left[b_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^j + \overline{b}_{-1}(a_{-1})^j(\overline{a}_{-1})^{j-1}\right]\;|0\rangle\,. \end{aligned} \tag{5.21}$$

More powers of J_- are going to bring in more of V's and $\overline{V}$'s, but will not bring in any $\overline{U}$'s. The lowest non-trivial state which contains a $\overline{U}$ is

$$\begin{aligned} |j;0,0;1,1\rangle = \frac{j}{2}\big[&(j-1)b^{\star}{}_{-1}b_{-1}(a_{-1})^{j-2}(\overline{a}_{-1})^j + jb^{\star}{}_{-1}\overline{b}_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^{j-1} - \\ &- a^{\star}{}_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^j + (j-1)\overline{b^{\star}}_{-1}\overline{b}_{-1}(\overline{a}_{-1})^{j-2}(a_{-1})^j + \\ &+ j\overline{b^{\star}}_{-1}b_{-1}(\overline{a}_{-1})^{j-1}(a_{-1})^{j-1} - \overline{a^{\star}}_{-1}(\overline{a}_{-1})^{j-1}(a_{-1})^j\big]\;|0\rangle\,. \end{aligned} \tag{5.22}$$

The operator structure for these terms are

$$\begin{aligned} |j;0,0;1,1\rangle = \frac{j}{2}\big[&(j-1)(\partial_z\overline{V})(\partial_z V)(\partial_z U)^{j-2}(\partial_{\bar{z}}U)^j + j(\partial_z\overline{V})(\partial_{\bar{z}}V)(\partial_z U)^{j-1}(\partial_{\bar{z}}U)^{j-1} - \\ &- (\partial_z\overline{U})(\partial_z U)^{j-1}(\partial_{\bar{z}}U)^j + (j-1)(\partial_{\bar{z}}\overline{V})(\partial_{\bar{z}}V)(\partial_{\bar{z}}U)^{j-2}(\partial_z U)^j + \\ &+ j(\partial_{\bar{z}}\overline{V})(\partial_z V)(\partial_{\bar{z}}U)^{j-1}(\partial_z U)^{j-1} - (\partial_{\bar{z}}\overline{U})(\partial_{\bar{z}}U)^{j-1}(\partial_z U)^j\big]\,. \end{aligned} \tag{5.23}$$

The term which involves $\overline{U}$ contains lower powers of j and would be subdominant. This becomes clear when we consider the action of J_+ on this state and see how it acts as an annihilation operator of a Heisenberg algebra in the large-j limit. One gets

$$\begin{aligned} J_+|j;0,0;1,1\rangle = -\frac{j}{2}\big[&(j-1)b_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^j + j\overline{b}_{-1}(a_{-1})^j(\overline{a}_{-1})^{j-1} + \\ &+ b_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^j + (j-1)\overline{b}_{-1}(a_{-1})^j(\overline{a}_{-1})^{j-1} + \\ &+ jb_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^j + b_{-1}(a_{-1})^j(\overline{a}_{-1})^{j-1}\big]\;|0\rangle\,. \end{aligned} \tag{5.24}$$

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Each term in (5.24) is the action of J_+ on the corresponding term in (5.23). The final result is as expected

$$J_+|j;0,0;1,1\rangle = 2j|j;0,0;0,1\rangle\,. \tag{5.25}$$

At this level, the action of J_+ is like that of a Heisenberg annihilation operator even for finite j. However the significant point is that in the large-j limit the terms which contain $a^\star{}_{-1}$ in (5.23) do not contribute. The situation becomes clear at the next level, viz. the action of J_+ on $|j;0,0;2,1\rangle$. Following similar manipulations we find that the corresponding operator which contains $\overline{U}$ does not contribute in the large-j limit.

The basic reason behind this may be found by examining the charges, eq. (5.4). It is clear from these expressions that $\overline{U}$ can be introduced only if there is a V or a $\overline{V}$ in the operator in an earlier stage, which may be replaced by $\overline{U}$. However each such term is accompanied by a term which replaces one of the U's by a V or a $\overline{V}$ - and there are lot more of these terms since the number of $V,\overline{V}$ are always much smaller than the number of U's.

We therefore conclude that the contraction of the rotation algebra to a Heisenberg algebra shows that the states which survive in this limit do not contain modes of the field $\overline{U}$.

This argument may be generalized to higher dimensions trivially. This is because we are working with an internal symmetry. Once again the major difference is that the fields U,V themselves are conformal fields and therefore the highest weight states are created by U^j itself. Consequently all the modes in this subsection would be modes with zero angular momentum rather than with angular momenta ± 1. Action of R symmetry generators of course do not change the angular momentum.

5.3 Combining the conformal and R symmetry parts

Let us now consider mixed descendants. The lowest such state is

$$\begin{aligned}|j;1,0;1,0\rangle \;=\; -j\,\big[&(j-1)a_{-2}b_{-1}(a_{-1})^{j-2}(\overline{a}_{-1})^j + ja_{-2}\overline{b}_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^{j-1} + \\ &+ b_{-2}b_{-1}(a_{-1})^{j-1}(\overline{a}_{-1})^j\big]\;|0\rangle\end{aligned} \tag{5.26}$$

the structure of the operator which creates this state is

$$\begin{aligned}-j\,\big[&(j-1)(\partial_z^2 U)(\partial_z V)(\partial_z U)^{j-2}(\partial_{\overline{z}} U)^j + j(\partial_z^2 U)(\partial_{\overline{z}} V)(\partial_z U)^{j-1}(\partial_{\overline{z}} U)^{j-1} + \\ &+ (\partial_z^2 V)(\partial_z V)(\partial_z U)^{j-1}(\partial_{\overline{z}} U)^j\big]\,.\end{aligned} \tag{5.27}$$

This is the first non-trivial operator which contains $\partial_z^2 V$. However it is clear that the term which involves $\partial_z V$ is subdominant in the large-j limit. One can now go ahead and examine the action of J_+ or L_+ on this state and verify that the expected answers follow. In all these steps the term which involves $\partial_z^2 V$ does not contribute in the large-j limit. A similar result follows for $\partial_z^2 \overline{V}$ when we consider the state $|j;1,0;0,1\rangle$. The situation becomes clearer at the next level, which we have checked, but will not present here.

Once again the reason behind this is clear from the generators. Higher angular momentum modes of $V,\overline{V}$ can be obtained only when L_- or $\overline{L}_-$ act. These are derivatives which get distributed over all the terms in the product, and since there are always lot more U's compared to V or $\overline{V}$'s, the dominant terms are those in which the derivatives act on U's.

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We therefore conclude that in this limit the fields V must be restricted to its lowest non-trivial angular momentum mode. Once again the argument generalizes to higher dimensions with the obvious differences noted above.

Let us therefore list the various facts which result from examining the contraction of the product of conformal and R symmetry groups to lower-dimensional rotation and R symmetry groups and Heisenberg groups. Starting with a state constructed from products of an elementary field U with SO(2) R charge +1

1. The field U is restricted to its two lowest non-trivial angular momentum states. (For $d > 2$ these are the zero mode and the d $l = 1$ modes.)
2. The field $\overline{U}$ does not appear in the operators which create the states
3. The fields which are neutral under the SO(2) are restricted to their lowest non-trivial angular momentum state.

These are crucial ingredients in the work of BMN [4]. We have obtained them from symmetry considerations.

5.4 States in the dual gauge theory

For the case of most interest, $AdS_5 \times S^5$ the dual CFT is a $\mathcal{N} = 4$, SU(N) gauge theory living on $S^3 \times \texttt{time}$. The conformal group is SO(4, 2) and the R-symmetry is SO(6). In the pp-wave description, the Penrose limit leads to $\mathrm{SO}(4) \times \mathrm{SO}(4) \times H(4) \times H(4)$ and according to the proposal of BMN is the states of string theory in the pp-wave background are a subset of states in the gauge theory with large R-charge J and large dimension Δ and small $\Delta - J$.

The considerations of the above subsections may be viewed as a carricature of this. We have not considered the effects of the gauge group and our considerations were restricted to free field theory.

The contraction of the conformal and the R-symmetry algebras does not depend on the fact that these symmetries act on gauge fields and adjoint scalars. The specific way these are represented, however, differ from the considerations of this section — the operators in question involve traces over the gauge group. Nevertheless we expect that the conclusions remain at the free field level. The essential point is that the operators which dominate in the large-j limit are those which are obtained when the lowering operators act on the string of U's rather than other fields which are brought down at earlier levels. This should lead to a restricted Hilbert space $\mathcal{H}_h \subset \mathcal{H}$ of states created by operators which do not contain higher angular modes of the elementary fields.

When we turn on interactions, one has to ask about transition amplitudes between states in the restricted hilbert state $\mathcal{H}_h$ and the states in $\mathcal{H} - \mathcal{H}_h$. In [4] it was argued that at large-N operators containing higher angular momentum modes acquire a large anomalous dimension and therefore decoouple from the theory, so that these transition amplitudes vanish. This is where large-N and large 't Hooft coupling enters into the discussion and is the reason why our conclusions regarding a contraction of the Hilbert space would be valid in the full theory.

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Usually the large-N limit reduces the allowed interactions to planar diagrams but do not reduce the free Hilbert space. On the other hand, as we have seen above, large conformal weight and large R-charge reduce the free Hilbert space without affecting the intensity of interaction. When we combine the two together we get a consistent truncation of the Hilbert space. Supersymmetry is a crucial ingredient in this since this ensures that amplitudes for transitions between states in the reduced space remain finite while those which take us out of this reduced space are suppressed. For non-supersymmetric theories like QCD, very likely the analog of the large-J limit contraction should be defined as an infinite momentum frame limit similar to the original definition of Matrix theory as the infinite momentum frame limit of eleven-dimensional supergravity.

5.5 A connection to the bulk

The bulk generators of the Heisenberg algebra, eq. (3.5) appear to be quite different from the representation in conformal field theory discussed in the previous section. Unlike $AdS \times S$ spacetimes the bulk generators do not reduce to the CFT generators when restricted to some region of the spacetime (e.g. the boundary). Is there any sense in which the CFT continues to provide a "holographic" description of the bulk?

In $AdS \times S$, the holographic representation of the AdS isometries is in terms of conformal symmetries of a CFT while the S isometries are represented as internal symmetries. When we take the Penrose limit both of these isometries becomes products of rotation group and Heisenberg group and there is a Z_2 symmetry between these. One would expect that the same phenomenon appears in a holographic description.

The above discussion shows that the CFT essentially reduces to a finite number of quantum mechanical degrees of freedom. For $d > 2$ these include the $d + 1$ zero modes of the scalars and d lowest angular momentum modes of the scalar U. The symmetries may the then written in terms of these modes and derivatives with respect to them. The generators of the Heisenberg algebra in the bulk involve derivatives with respect to the transverse coordinates to the pp-wave. In [5] it was proposed that this implies that there should a holographic representation in d euclidean dimensions. In view of the above results it is tempting to attempt a different interpretation, viz. the transverse coordinates in the bulk should be thought of as these $2d$ quantum mechanical degrees of freedom. It appears, however that the CFT furnishes a Bargmann-Fock representation of the harmonic oscillator algebra while the bulk furnishes a usual coordinate representation.[3] This would make the Z_2 symmetry more manifest. The details of this correspondence, especially the role of the gauge field remain to be understood.

6. Correlators in the large R charge limit

In the standard AdS/CFT correspondence the n-point correlators of conformal fields are related to bulk correlators via boundary-to-bulk Green's functions [17]. Consider for example the two point function of an operator $\mathcal{O}_{h,n_i}$ which has vanishing anomalous dimension.

[3]A related point has been made in Arutyunov and Sokatchev, [16]

The lowest energy state in the CFT defined on S^{d-1} created by this operator are the ones which are obtained from a primary state by the action of d raising operators n_i times. The energy of this state is $h + \sum_i n_i$. This is also the conformal dimension of the operator, so that on R^d the two point function is given by

$$\langle \mathcal{O}_{h,n_i}(x)\mathcal{O}_{h,n_i}(0)\rangle = \frac{1}{|x|^{2h+2\mathcal{N}}}\,, \tag{6.1}$$

where $\mathcal{N} = \sum_i n_i$. Here $|x|$ denotes a radial coordinate in the R^d. This is related to the euclidean time coordinate τ of the CFT defined on a S^{d-1} by $|x| = e^\tau$. Since this is the global time in the AdS we get the relationship

$$\langle h, n_i|e^{-\tau H}|h, n_i\rangle = \frac{1}{|x|^{2h+2\mathcal{N}}}\,, \tag{6.2}$$

where H is the hamiltonian of the bulk theory.

In the strict $h = \infty$ limit the above correlator is zero for $|x| > 1$ and infinite for $|x| < 1$. To make concrete connections to bulk quantities it is first necessary to define finite quantities starting from these correlation functions.

We have seen that in the Penrose limit the conformal and R symmetry algebras reduce to Heisenberg algebras and we can write

$$H = C + P\,, \tag{6.3}$$

where C denotes the common central extension and P denotes the outer automorphism. In the bulk we have $C \sim p_- = p^+$ and $P = p_+ = p^-$. This suggests that we *define* normalized two point functions

$$G(|x|) = \frac{\langle h, n_i|e^{-\tau H}|h, n_i\rangle}{\langle h, 0|e^{-\tau H}|h, 0\rangle}\,. \tag{6.4}$$

Note that in the bulk the state $|h, 0\rangle$ is the light cone vacuum with $p_- = h$ Then the above correspondence implies

$$G(|x|) = \langle h, n_i|e^{-\tau P^-}|h, n_i\rangle\,. \tag{6.5}$$

We will now consider the large-J limit of CFT three point functions. As is well known, in CFT the three point function is completely determined by the OPE

$$O_i(x)O_j(0) = C_{ijk}\frac{1}{|x|^{(h_k-h_i-h_j)}}O_k(x)\,. \tag{6.6}$$

Let us now assume that the conformal weights are such that $h_i = J_i + p_i$ for p_i small but finite numbers and where J_i represent U(1) charges. We are interested in the limit of the OPE for large J_i. Conservation of U(1) charge $J_k = J_i + J_j$ implies that the space time dependence of the OPE is simply $1/|x|^{(p_k-p_i-p_j)}$ i.e. it is independent of J. However the structure constants C_{ijk} will generically depend on the values of the U(1) charges J_i. For instance in the case of BMN operators we have $C_{ijk} = \sqrt{J_iJ_jJ_k}/N$. In the BMN limit where $J \sim \sqrt{N}$, the structure constant scale like $1/\sqrt{J}$ and therefore goes to zero in the limit $J = \infty$. This motivates a redefinition of the structure constants to have a finite OPE in the large-J limit. In what follows we suggest a way to do it. Let us first write the OPE as

$$C_{ijk} \, [1 - (p_k - p_i - p_j) \; \log|x|] \; O_k(0) \tag{6.7}$$

and let us define a finite double limit, $J \to \infty$ and $|x| \to 0$ with $y = \log|x|/\sqrt{J}$ held finite. In other words as we increase J we focus on smaller values of $|x|$. Now we have

$$C_{ijk} \; \log|x| = \frac{a^2}{\sqrt{J}} \log(|x|) \,, \tag{6.8}$$

where $a^2 = J^2/N$. The OPE can be now represented as a vertex operator, namely

$$\langle i, j, k|V\rangle = y \cdot a^2 \cdot (p_k - p_i - p_j) \tag{6.9}$$

with the new "blow up" variable y representing now the interaction amplitude. This structure of the vertex operator was already suggested in ref. [19]. In summary, we simply observe that in the large-J limit the main contribution to the OPE comes from the contact terms appearing in the limit $|x - y| = 0$ with the corresponding pole being compensated by the zero of the structure constant in the large-J limit. It is natural to conjecture that in the large-J limit generic correlators could be mapped into string like diagrams defined in terms of the vertex opertor (6.9) and the free propagators (6.5). For recent discussions on the three point function see refs. [8, 20].

7. Conclusions

We have studied how the conformal and R-symmetry algebras in a CFT contract to products of lower-dimensional rotation and R symmetry algebras and Heisenberg algebras. This parallels the similar contraction in the Penrose limit of $AdS \times S$ spacetimes. We have argued that in this contraction process, higher angular momentum modes of fields in the CFT decouple and either the field with SO(2) R-charge +1 or with SO(2) R-charge −1 remain (but not both). We have not considered the supersymmetries; we expect that analogous considerations for the superconformal algebras will provide more insight.

The fact that at large weight and large R charge one gets Heisenberg algebras is quite general. We have explicitly worked out how this happens for the $d = 2$ case, but it is clear that the result generalizes to any number of dimensions. The reduction of the Hilbert space has been, however, demonstrated in a toy model of $1 + 1$-dimensional CFT which is a recognizable carricature of higher-dimensional models. This makes it quite plausible that one could generalize the considerations to higher dimensions. However several crucial ingredients are absent in our toy model — these are related to the fact that in higher dimensions the CFT is a gauge theory and one has to perform a large-N limit of this gauge theory. It is important to investigate whether such symmetry considerations may be used to understand the phenomenon in gauge theory — this would also elucidate the role of large-N limit. The intricacies of operator structure and operator mixing [21] would be relevant to this.

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In this paper we have dealt with supergravity modes in the bulk and their CFT descriptions. These are the states which are created using the isometries of the geometry. The most interesting feature of the pp wave — CFT correspondence is, however the fact that a string theory is tractable in this geometry and [4] have found how to describe the higher stringy modes in the gauge theory. We have not dealt with such stringy modes in this paper. However it is reasonable to believe that one can understand the spectroscopy of stringy modes in terms of worldsheet current algebras and one has to understand how this is realized in the CFT. A requirement that the CFT encodes this current algebra correctly should throw light on the proposal of [4] for CFT description of stringy states.

Finally, the meaning of holography in this correspondence remains unclear. The present paper reinforces the claim that a large R-charge limit of the original CFT is dual to the bulk theory. But in what sense is this a holographic description? The dual CFT lives on a $S^{d-1} \times \texttt{time}$ which is the boundary of the AdS_{d+1} spacetime *before* any Penrose limit. The Penrose limit, on the contrary, focusses on the deep interior of the AdS which is not a part of the pp-wave geometry. The fact that the CFT essentially becomes a quantum mechanical system might suggest, however, that it is natural to place this on the one-dimensional boundary of pp-wave — as suggested in [8]. However, we do not have any concrete check of this idea yet. What does appear to be true is that all the symmetries of the bulk are realized as internal symmetries of this effective quantum mechanics. It appears likely that for the eleven-dimensional pp-wave the theory on the boundary is the Matrix theory in this background written down in [4] and further studied in [22]. It would be interesting to see if this can be made concrete.

The key point to understand is whether the scale of the CFT appears as some coordinate in the pp-wave in a way similar to what happened in the AdS/CFT correspondence. One possible scenario is that *worldsheet dilatations* become a holographic coordinate, in a way similar to non-critical string theory [23, 24]. The CFT would be then similar to the $c = 1$ matrix model [25]. If this is true, one should be able to recast the contracted form of the beta function equations of the CFT as worldsheet beta functions which would be related to bulk equations of motion in the usual way. For a recent discussion of RG flows in this context see [26].

Acknowledgments

S.R.D. would like to thank S. Mathur, S. Minwalla, Y. Nambu and especially A. Jevicki for discussions. C.G. would like to thank A. Uranga for discussions. The work of S.R.D. is partially supported by U.S. DOE contract DE-FG01-00ER45832. The work of C.G. is partially supported by grant AEN2000-1584.

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